# Blow-up profile for a nonlinear heat equation with the Neumann boundary condition (Evolution Equations and Asymptotic Analysis of Solutions)



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数理解析研究所講究録 1358 巻 2004 年 110-116



# Blow-up profile for a nonlinear heat equation with the Neumann boundary condition

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October 18, 2003

This paper is concerned with the nonlinear diffusion equation

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \end{cases}$$

where $\Omega$ is a bounded smooth domain in $\mathbf{R}^N$, $\nu$ is the unit outward normal vector on $\partial\Omega$, $p > 1$ is a constant and $u_0 \in L^\infty(\Omega)$ is a nonnegative function with $\|u_0\|_\infty \neq 0$. For the solution $u(x,t)$ of the nonlinear diffusion equation, the *blow-up time* $T$ is defined by

$$T = \sup\{\tau > 0 \mid u(x,t) \text{ is bounded in } \bar{\Omega} \times (0,\tau)\}.$$

Then, $0 < T < +\infty$ and $\overline{\lim}_{t\to T}\|u(x,t)\|_{C(\bar{\Omega})} = +\infty$ hold. The *blow-up set* of the solution $u(x,t)$ is defined as the set

$$\{x \in \bar{\Omega} \mid \text{there is a sequence } (x_n,t_n) \text{ in } \bar{\Omega} \times (0,T) \text{ such that}$$

$$(x_n,t_n) \to (x,T) \text{ and } u(x_n,t_n) \to +\infty \text{ as } n \to \infty\}.$$

This set is a nonempty closed set in $\bar{\Omega}$. From standard parabolic estimates, we can obtain the *blow-up profile*, which is a continuous function defined by

$$u_*(x) = \lim_{t\to T} u(x,t)$$

outside the blow-up set.

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The blow-up problem has been studied by many authors since the pioneering work due to Fujita [13]. There are a number of results for the nature of the blow-up set. For the Cauchy problem with $(N-2)p < N+2$, Velázquez [34] showed that the $(N-1)$-dimensional Hausdorff measure of the blow-up set is bounded in compact sets of $\mathbf{R}^N$ whenever the solution is not the constant blow-up one $(p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}$. For the Cauchy problem or the Cauchy-Dirichlet problem in a convex domain with $(N-2)p < N+2$, Merle and Zaag [25] showed that for any finite set $D \subset \Omega$, there exists $u_0$ such that the blow-up set is $D$ (See also [1] and [3]). For the Cauchy problem with $N = 1$, Herrero and Velázquez [17] showed that for any point $\bar{x}$ in the blow-up set of a solution $\bar{u}$ and $\varepsilon > 0$, there exists $u_0$ with $\|u_0 - \bar{u}_0\|_C \leq \varepsilon$ such that the blow-up set of $u$ consists of a single point $x$ with $|x - \bar{x}| \leq \varepsilon$. For the Cauchy-Dirichlet problem in an ellipsoid centred at the origin with $(N-2)p < N$, Filippas and Merle [10] showed that if the blow-up time is large, then the blow-up set consists of a single point near the origin. Also, for the Cauchy or Cauchy-Dirichlet problem with $(N-2)p < N+2$, the second author [27] showed the following. For any nonnegative function $\phi \in C(\bar{\Omega})$ and $\delta > 0$, if $\varepsilon > 0$ is small, then any point $x$ in the blow-up set satisfies $\phi(x) \geq \max_y \phi(y) - \delta$ for $u_0 = \varepsilon^{-1}\phi$. For the Cauchy-Neumann problem, the first author [18] showed the following. Suppose that $\Omega = (0,\pi) \times \Omega_0$ is a cylindrical domain with a bounded smooth domain $\Omega_0$ in $\mathbf{R}^{N-1}$ and that a nonnegative function $\phi \in L^\infty(\Omega)$ satisfies $\int_\Omega \phi(x_1, x_2, \cdots, x_N) \cos x_1 dx > 0$. If $\varepsilon > 0$ is small, then the blow-up set is contained in the base plane $\{0\} \times \bar{\Omega}_0$ for $u_0 = \varepsilon\phi$. Recently, for the Cauchy-Neumann problem with $(N-2)p < N+2$, the first and second authors [20] obtained the following. Let $P$ be the orthogonal projection in $L^2(\Omega)$ onto the eigenspace corresponding to the second eigenvalue of the Laplace operator with the Neumann condition. For any nonnegative function $\phi \in L^\infty(\Omega)$ and neighborhood $W$ of $\{x \in \bar{\Omega} \mid (P\phi)(x) = \max_{y\in\bar{\Omega}}(P\phi)(y)\} \cup \partial\Omega$, if $\varepsilon > 0$ is small, then the blow-up set is contained in $W$ for $u_0 = \varepsilon\phi$. See, e.g., the references in this paper for related results or other studies on blow-up formation in $u_t = \Delta u + u^p$.

In this papar, we study the blow-up profile.

For large initial data $u_0^\varepsilon = \varepsilon^{-1}\phi$, we have the following.

**Theorem 1** ([35]) *Let $\phi \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the blow-up set of the solution $u^\varepsilon$ with the initial data $u_0^\varepsilon = \varepsilon^{-1}\phi$ is contained in the set $S := \{x \in \bar{\Omega} \,|\, \phi(x) \geq \max_{y\in\bar{\Omega}} \phi(y) - \delta\}$ and the blow-up profile $u_*^\varepsilon$ satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(\phi(x)^{-(p-1)} - (\max_{y\in\bar{\Omega}}\phi(y))^{-(p-1)}\right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Theorems 2 and 3 are instability results for constant blow-up solutions.

**Theorem 2** ([36]) *Let $f \in C(\bar{\Omega})$ be a positive function, and let $\delta$ and $T_0$ be positive constants. Then, there exist $C$ and $\varepsilon_0 > 0$ satisfying the following: For any $\varepsilon \in (0, \varepsilon_0]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ satisfying $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ and*

$$\left\| u_0^\varepsilon(x) - (p-1)^{-\frac{1}{p-1}} T_0^{-\frac{1}{p-1}} \right\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$$

*such that the blow-up time of the solution $u^\varepsilon$ with initial data $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ is larger than $T_0$ and the inequality*

$$\|\varepsilon u^\varepsilon(x, T_0) - f(x)\|_{C(\bar{\Omega})} \leq \delta$$

*holds.*

**Theorem 3** ([36]) *Let $f \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$, and let $\delta$ and $c$ be positive constants. Then, there exist $C$ and $\varepsilon_0 > 0$ satisfying the following: For any $\varepsilon \in (0, \varepsilon_0]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ with $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ and $\|u_0^\varepsilon - c\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$ such that the blow-up set of the solution $u^\varepsilon$ with the initial data $u_0^\varepsilon$ is contained in the set $S := \{x \in \bar{\Omega} \,|\, f(x) \geq \max_{y\in\bar{\Omega}} f(y) - \delta\}$ and the blow-up profile $u_*^\varepsilon$ satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(f(x)^{-(p-1)} - (\max_{y\in\bar{\Omega}} f(y))^{-(p-1)}\right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Let $\lambda_i$ be the $i$-th eigenvalue of $-\Delta\varphi = \lambda\varphi$ with the Neumann boundary condition $\frac{\partial\varphi}{\partial\nu} = 0$, where $0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. We denote the orthogonal projection in $L^2(\Omega)$ onto the eigenspace $X_i$ corresponding to the $i$-th eigenvalue by $P_i$. Here, we remark that $P_1\phi = \frac{1}{|\Omega|}\int_\Omega \phi\, dx$ is a constant.

For small initial data $u_0^\varepsilon = \varepsilon\phi$, the first and second authors already showed Propositions 4 and 5 below.

**Proposition 4** ([20]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$. Then, there exist a constant $\varepsilon_0 > 0$ and a family $\{(t^\varepsilon, \delta^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]} \subset \mathbf{R}^2$ such that the solution $u^\varepsilon$ with the initial data $u_0^\varepsilon = \varepsilon\phi$ and its blow-up time $T^\varepsilon$ satisfy $\lim_{\varepsilon\to+0} t^\varepsilon = 1$, $\lim_{\varepsilon\to+0} \varepsilon^{p-1}T^\varepsilon = (p-1)^{-1}(P_1\phi)^{-(p-1)}$, $\lim_{\varepsilon\to+0} \varepsilon^{p-1}e^{\lambda_2 T^\varepsilon}\delta^\varepsilon = (p-1)^{-1}(P_1\phi)^{-p}$ and*

$$\lim_{\varepsilon\to+0} \Bigg\| \frac{t^\varepsilon}{\delta^\varepsilon}\left(1 - (p-1)^{\frac{1}{p-1}} t^{\varepsilon\frac{1}{p-1}} u^\varepsilon(x, T^\varepsilon - 1)\right)$$

$$-e^{\lambda_2}\left((\max_{y\in\bar\Omega}(P_2\phi)(y)) - (P_2\phi)(x)\right)\Bigg\|_{L^\infty(\Omega)} = 0.$$

**Proposition 5** ([19]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$. Then, there exist $C$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the solution $u^\varepsilon$ with the initial data $u_0^\varepsilon = \varepsilon\phi$ and its blow-up time $T^\varepsilon$ satisfy $u^\varepsilon(x,t) \leq C(T^\varepsilon - t)^{-\frac{1}{p-1}}$ for all $(x,t) \in \bar\Omega \times [T^\varepsilon - 1, T^\varepsilon)$.*

We obtain the following as a corollary of the propositions above.

**Theorem 6** ([21]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the blow-up set of the solution $u^\varepsilon$ with the initial data $u_0^\varepsilon = \varepsilon\phi$ is contained in the set $S := \{x \in \bar\Omega |\, (P_2\phi)(x) \geq \max_{y\in\bar\Omega}(P_2\phi)(y) - \delta\}$. Further, the blow-up time $T^\varepsilon$ and the blow-up profile $u_*^\varepsilon$ satisfy the inequality*

$$\left|\varepsilon^{p-1}T^\varepsilon - (p-1)^{-1}(P_1\phi)^{-(p-1)}\right| + \Bigg\|\varepsilon^{-1}e^{-\frac{\lambda_2 T^\varepsilon}{p-1}}u_*^\varepsilon(x)$$

$$-(p-1)^{-\frac{1}{p-1}}(P_1\phi)^{\frac{p}{p-1}}\left((\max_{y\in\bar\Omega}(P_2\phi)(y)) - (P_2\phi)(x)\right)^{-\frac{1}{p-1}}\Bigg\|_{C(\bar\Omega\setminus S)} \leq \delta.$$

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